TIME AND FREQUENCY DOMAIN RELATIONSHIPS FOR SECOND ORDER SYSTEMS

There are many reasons for considering the time and frequency domain relationships of a second order system in the study of operational amplifiers. One is that many operational amplifier configurations can be modeled with reasonable accuracy assuming just a second order system. Such a procedure represents a reasonable compromise between complexity and accuracy of the model. Another reason is that these relationships allow us to predict frequency domain performance from the simpler to measure time domain performance.

General Second Order System in the Frequency Domain

The general transfer function of a low-pass, second order system in the frequency domain using voltage variables is

$$A(s) = \frac{V_o(s)}{V_{in}(s)} = \pm \frac{A_o\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \pm \frac{A_o\omega_o^2}{s^2 + (\omega_o/Q)s + \omega_o^2} \qquad (1)$$

where A_o = the low frequency gain of $V_o(s)/V_{in}(s)$

$\omega_o = \omega_n$ = the pole frequency in radians per second

ζ = the damping factor (=1/2Q)

Q = the pole Q (=1/2ζ)

Equation (1) is illustrated in Fig. 1.

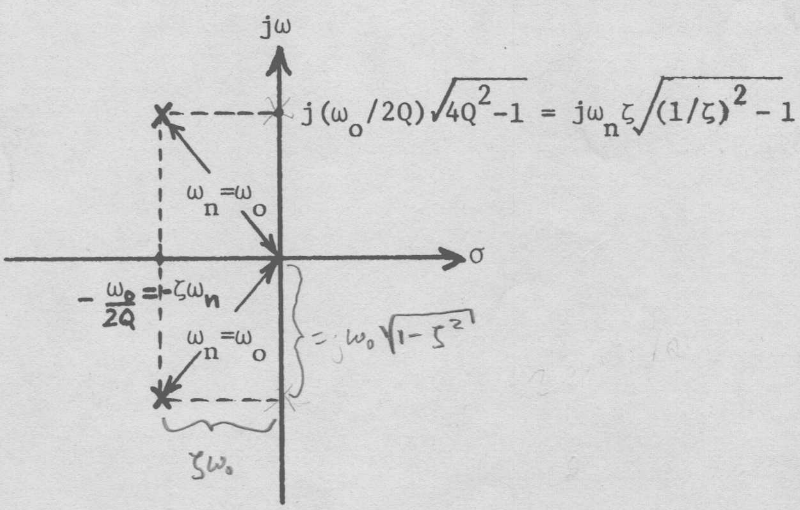


Fig. 1

The magnitude of the frequency response can be found from (1) as

$$|A(j\omega)| = \frac{A_o \omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2}} \qquad (2)$$

However, (2) may be generalized by normalizing the amplitude with respect to A_o and the radian frequency by ω_n to give

$$\frac{|A(j\omega/\omega_n)|}{A_o} = \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + 4\zeta^2 (\omega/\omega_n)^2}} \qquad (3)$$

A plot of (3) in dB versus log ω is shown in Fig. 2 where ζ or 1/2Q is used as a parameter.

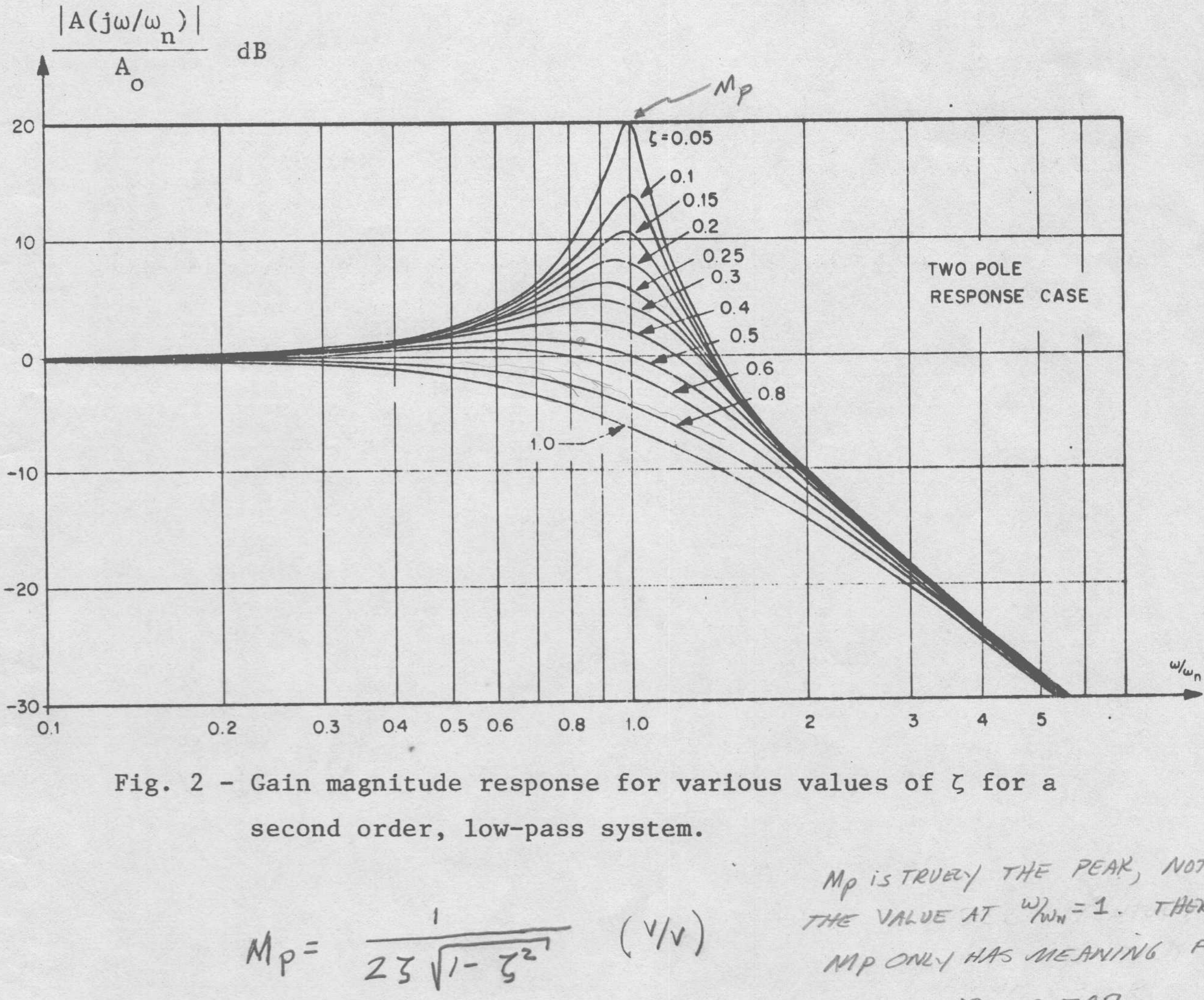


Fig. 2 - Gain magnitude response for various values of ζ for a second order, low-pass system.

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad (V/V)$$

M_p IS TRUELY THE PEAK, NOT THE VALUE AT $\omega/\omega_n = 1$. THEREFORE, M_p ONLY HAS MEANING FOR $\zeta < 0.707$

$$M_{p(dB)} = 20 \log\left[\frac{1}{2\zeta\sqrt{1-\zeta^2}}\right]$$

SEE TABLE AT THE END OF THIS BOOK

The second order function of (1) is found in the analysis of many practical systems. Consider the single-loop, feedback block diagram of Fig. 3.

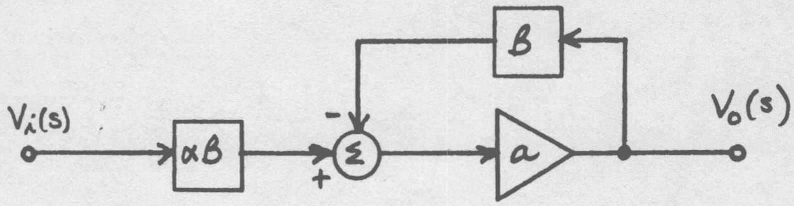


Fig. 3 - Single-loop, feedback block diagram.

The closed loop gain , A(s), can be expressed as

$$A(s) = \frac{V_o(s)}{V_i(s)} = \frac{\alpha a\beta}{1+a\beta} \tag{4}$$

Let us assume that α and β are real and that "a" is the amplifier's gain and can be approximated as

$$a(s) \simeq \frac{a_o\omega_1\omega_2}{(s+\omega_1)(s+\omega_2)} \tag{5}$$

where a_o is the DC gain of the amplifier and ω_1 and ω_2 are real axis poles. Substitution of (5) into (4) gives

$$A(s) = (\alpha\beta)\frac{a_o\omega_1\omega_2}{s^2+(\omega_1+\omega_2)s+\omega_1\omega_2(1+a_o\beta)} \tag{6}$$

Comparing (6) with (1) results in the following identifications.

$$A_o = (\alpha a_o\beta)(1+a_o\beta)^{-1} \tag{7}$$

$$\omega_n = \omega_o = \sqrt{\omega_1\omega_2(1+a_o\beta)} \tag{8}$$

and

$$2\zeta = 1/Q = \frac{\omega_1+\omega_2}{\sqrt{\omega_1\omega_2(1+a_o\beta)}} \tag{9}$$

The same principles can be applied to a second order band-pass or high-pass system, but the low-pass case is of more practical interest to us and will be the only one considered. It is also possible for β (and thus α) to become frequency dependent which further complicates the analysis.

Low-Pass Second Order System in the Time Domain

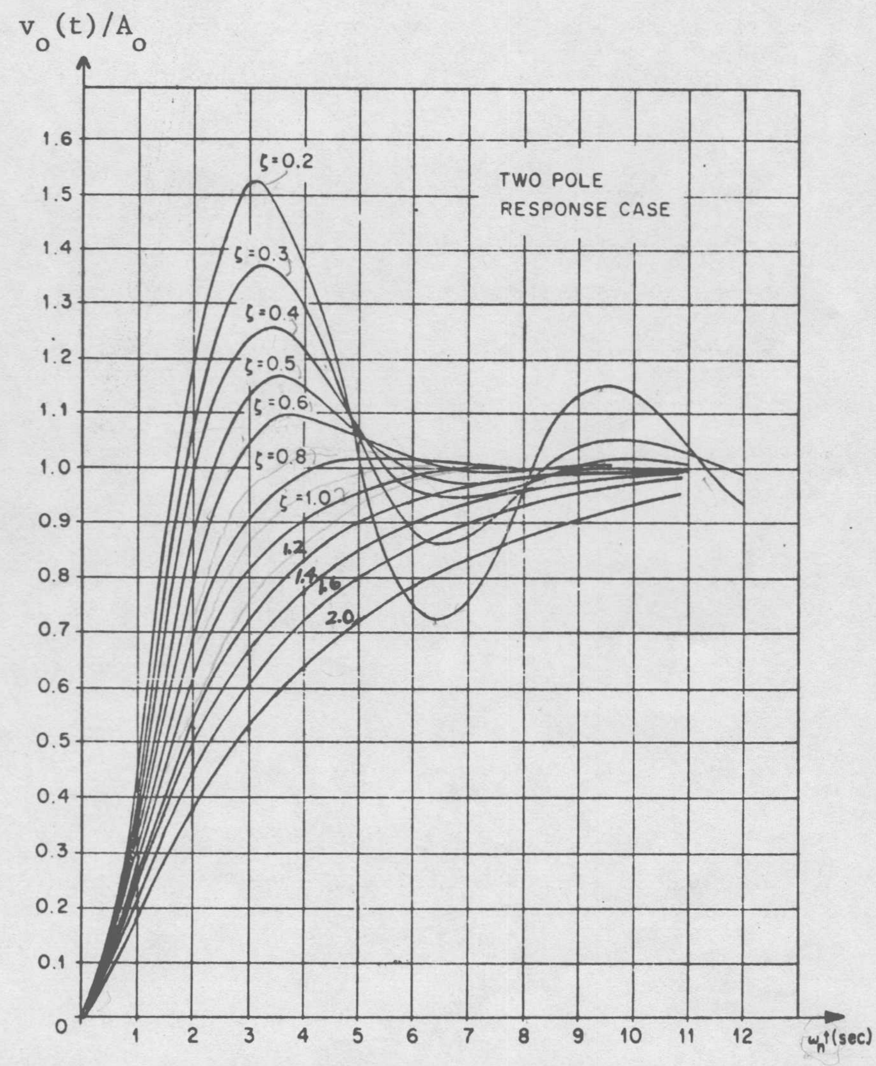
Unfortunately, it is time consuming to make measurements in the frequency domain. Therefore we are interested in determining the frequency domain performance from the time domain. This information is developed as follows. The general response of (1) to a unit step can be written as

$$v_o(t) = A_o[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\sqrt{1-\zeta^2}\,\omega_n t + \phi)] \tag{10}$$

where

$$\phi = \tan^{-1}[\frac{\sqrt{1-\zeta^2}}{\zeta}] \tag{11}$$

The step response plotted in normalized amplitude versus radians is shown in Fig. 4.



$Q = \frac{1}{2\zeta}$

$\zeta = 1$ for $Q = 0.5$

Fig. 4 - Step response as a function of ζ for a low-pass, 2nd order system.

Let us consider first the underdamped case where $\zeta<1$. For the underdamped case there will always be an overshoot as defined by Fig. 5.

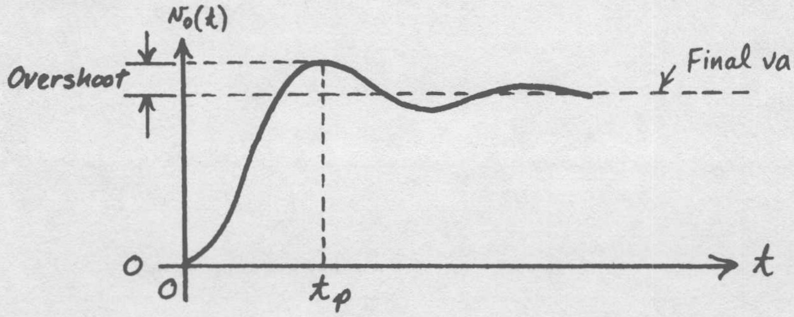


Fig. 5 - Illustration of overshoot and t_p.

o.s. is overshoot

The overshoot can be expressed as

$$\text{Overshoot} = \exp\left[\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right] \;;\; \zeta = \left[\frac{\ln^2(\text{O.S.})}{\pi^2 + \ln^2(\text{O.S.})}\right]^{1/2} \tag{12}$$

The time at which the overshoot occurs, t_p, is shown in Fig. 5 and can be found as

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{13}$$

Thus the measurement of the overshoot permits the calculation of ζ (or 1/2Q). With this information and the measurement of t_p, one can calculate ω_n from (13). Therefore the frequency response of a second order, low-pass system with $\zeta<1$ can be determined by measuring the overshoot and t_p of the step response.

Next consider the overdamped case where $\zeta\geq1$. In this case there is no overshoot. The unit step response can be simplified from (10) to yield

$$v_o(t) = A_o\left[1 - \frac{1}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-\omega_n t(\zeta-\sqrt{\zeta^2-1})}}{\zeta-\sqrt{\zeta^2-1}} - \frac{e^{-\omega_n t(\zeta+\sqrt{\zeta^2-1})}}{\zeta+\sqrt{\zeta^2-1}}\right)\right] \tag{14}$$

It is difficult to measure various aspects of this response and thus determine ζ and ω_n. Fortunately there are very few occasions where we have $\zeta>1$. If $\zeta>1$, then the best result is probably obtained by matching the step response to one of the curves for $\zeta>1$ of Fig. 4. More accuracy could be achieved by evaluating $v_o(t)$ for say $\omega_n t=4$ and selecting values of ζ until $v_o(4/\omega_n)$ matches with the experimental data at this point.

Determination of Phase Margin and Cross-over Frequency from ζ and ω_n

In the previous section we have seen how ζ and ω_n of a second order system can be determined by the time domain step response. It is the objective of this section to show how to find the phase margin, ϕ_m, and the cross-over frequency, ω_c, from ζ and ω_n. Fig. 6 shows the meaning of ϕ_m and ω_c.

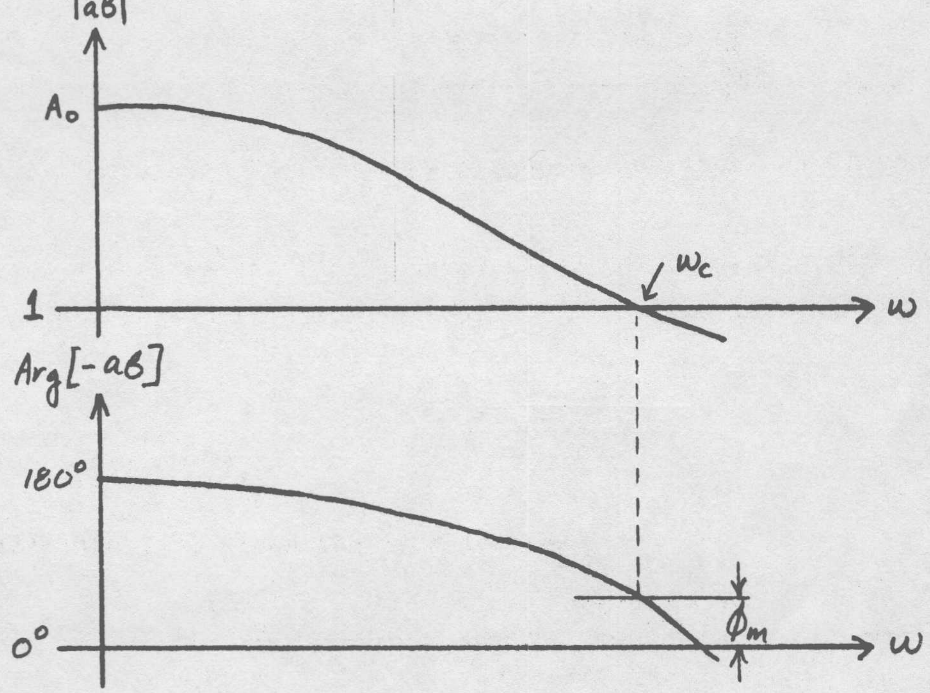


Fig. 6 - Illustration of cross-over frequency and phase margin.

In order to assist in developing the desired relationships, it will be convenient to assume that β (thus α) is real. From (4) we may solve for $a\beta$ to get

$$a\beta = \frac{1/\alpha}{(1/A) - (1/\alpha)} \tag{15}$$

Substituting (1) into (15) gives the loop gain as

$$a\beta = \frac{A_o\omega_n^2/\alpha}{(s^2 + 2\zeta\omega_n s + \omega_n^2) - (A_o\omega_n^2/\alpha)} = \frac{A_o/\alpha}{(\frac{s}{\omega_n})^2 + 2\zeta(\frac{s}{\omega_n}) + 1 - (A_o/\alpha)} \tag{16}$$

when $|\alpha\beta|=1$, then $\omega=\omega_c$ so that (16) becomes

$$|\alpha\beta| = \frac{A_o/\alpha}{\sqrt{[(1-\frac{A_o}{\alpha}) - (\omega_c/\omega_n)^2]^2 + [2\zeta(\omega_c/\omega_n)]^2}} \tag{17}$$

Since $|\alpha\beta|=1$, we may solve (17) for ω_c to get

$$\omega_c = \omega_n[\sqrt{[2\zeta^2 - (1-A_o/\alpha)]^2 - (1-2A_o/\alpha)} - 2\zeta^2 + (1-A_o/\alpha)]^{1/2} \tag{18}$$

Thus, knowing A_o and α and ω_n and ζ, we may calculate the cutoff frequency of a second order system. In an operational amplifier circuit, $\alpha=A_o$ so that (18) becomes

$$\omega_c = \omega_n[\sqrt{4\zeta^4+1} - 2\zeta^2]^{1/2} \tag{19}$$

Fig. 7 gives a plot of this useful function.

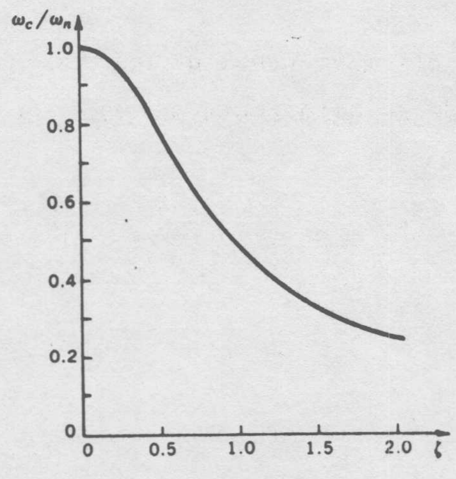


Fig. 7 - Plot of ω_c/ω_n versus ζ for a low-pass, second-order system.

The phase shift of $a\beta$ can be found from (16). However we must add $\pm\pi$ to this value to account for the minus sign of the summing junction of Fig. 3. Thus

$$\phi_m = -\tan^{-1}[\frac{2\zeta\omega_c/\omega_n}{(1-A_o/\alpha) - (\omega_c/\omega_n)^2}] \tag{20}$$

Since $A_o = \alpha$, we may write (20) as

$$\phi_m = \tan^{-1}\left[\frac{2\zeta}{\omega_c/\omega_n}\right] \tag{21}$$

Substituting (19) into yields

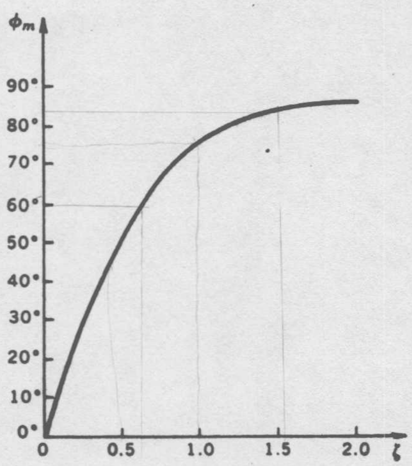
$$\phi_m = \tan^{-1}\left[\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right] \tag{22}$$

Another form of (22) which is equivalent is

$$\phi_m = \cos^{-1}\left[\sqrt{4\zeta^4+1} - 2\zeta^2\right] \tag{23}$$

Fig. 8 give a plot of ϕ_m of (22) or (23).

FOR $\zeta = 1$
$\phi_m = 76^\circ$



ζ	ϕ_m
0.42	45°
0.48	50°
0.54	55°
0.61	60°
0.70	65°
0.80	70°
0.95	75°

Fig. 8 - Plot of ϕ_m versus ζ for a low-pass, second-order system.

COMPENSATION OF OP AMPS

GENERAL

Compensation is necessary in op amp circuits because the very high values of the loop gain may lead to undesirable performance or instability depending on the feedback network. The basic single loop feedback diagram shown in Fig. 1 will be used to model the circuits discussed.

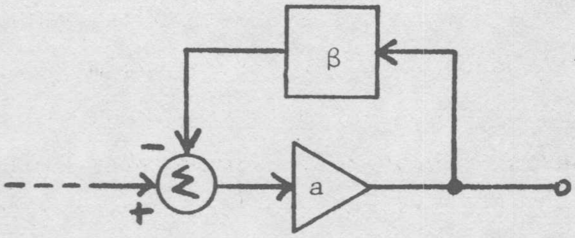


Fig. 1 - Block diagram of a single loop feedback system.

Notationwise, a is the forward, unloaded gain of the op amp and β is the reverse transmission factor. A very important consideration to be made in the analysis of the open loop gain, aβ, is that a is *considered large but not infinite.* Thus in the open loop considerations made in this analysis we will not be able to use the null port concept. This results in a unique circumstance which permits us to *place components in the open loop path which will effect the open loop gain but not effect the closed loop gain.* Let us expand on this very important principle further by considering the inverting and noninverting amplifiers of Fig. 2.

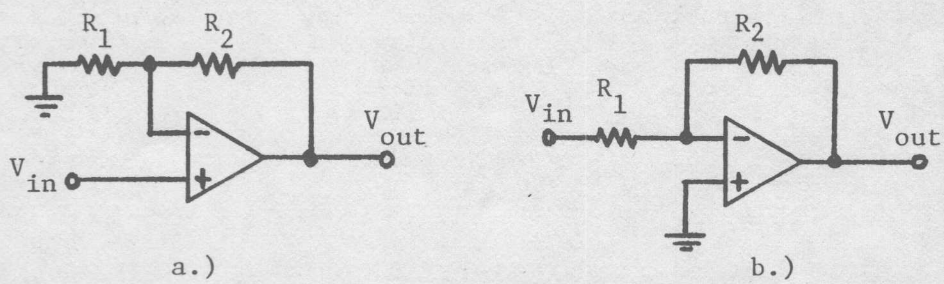


Fig. 2 - a.) Noninverting amplifier and b.) inverting amplifier configurations of the op amp.

The loop can be opened either at the input or the output of the op amp. If we choose the output, then Fig. 3 results for *both* of the configurations of Fig. 2. Remember that the signal source if a voltage is grounded so that the *open loop calculations for the noninverting and inverting configurations are identical.*

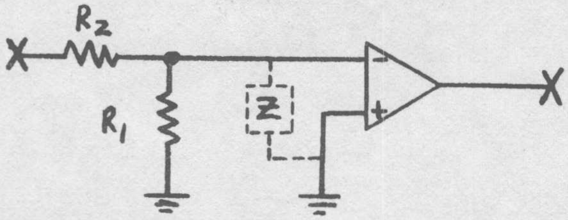


Fig. 3 - Open loop circuit of Fig. 2.

Now using the above principles, we can place an impedance, Z, as shown in Fig. 3 connected between the plus and minus input terminals of the op amp. This impedance will effect the open loop gain and in particular it will modify β, however, Z will not effect the closed loop gain of either circuit of Fig. 2 (unless the frequency is sufficiently high that aβ is approaching unity). Consequently, in our compensation schemes we shall try to add elements in the position of Z so that we may modify the loop gain but not effect the closed loop gain.

The approach to compensating op amp circuits can be divided into two methods depending upon whether the compensation is applied to the op amp, i.e. a, or whether the compensation is applied to the β network such as the Z in Fig. 3. We shall only consider the second method but it is worthwhile to mention that the familiar Miller compensation technique is an example of the first method. Obviously, internally compensated op amps such as the 741 will not require other compensation if the feedback network consists only of resistors. However, in many applications, active elements appear in the feedback loop and consequently, further compensation of internally compensated amplifiers may be required. As far as the user is concerned, this additional compensation will have to be of the second method which is the subject of this analysis.

The second method of compensation called β compensation has three basic techniques which are:

1. DC loop reduction.
2. Lag compensation.
3. Lead compensation.

These methods will be presented and explained with respect to the op amp circuits of Fig. 2.

DC LOOP REDUCTION

In the DC loop reduction method of compensation, the impedance Z in Fig. 3 is replaced by a resistor R. The application of the method is very straightforward as will be illustrated. The first step in all compensation schemes is to plot the magnitude and phase of the frequency response of the open loop gain, a . It helps to do this step twice. The first time the frequency includes the entire range of interest. Fig. 4 is an illustration of these plots. Notice that when $|a\beta| = 1$ (or 0 dB) that this defines the crossover frequency, ω_c. The phase margin, PM, of the system is then found by the difference between the phase plot and 0 degrees at the frequency ω_c. The second plot is the same as the first except the range in the vicinity of ω_c is expanded both in amplitude and frequency. For example, Fig. 5 represents a more detailed plot of Fig. 4 in the vicinity of ω_c.

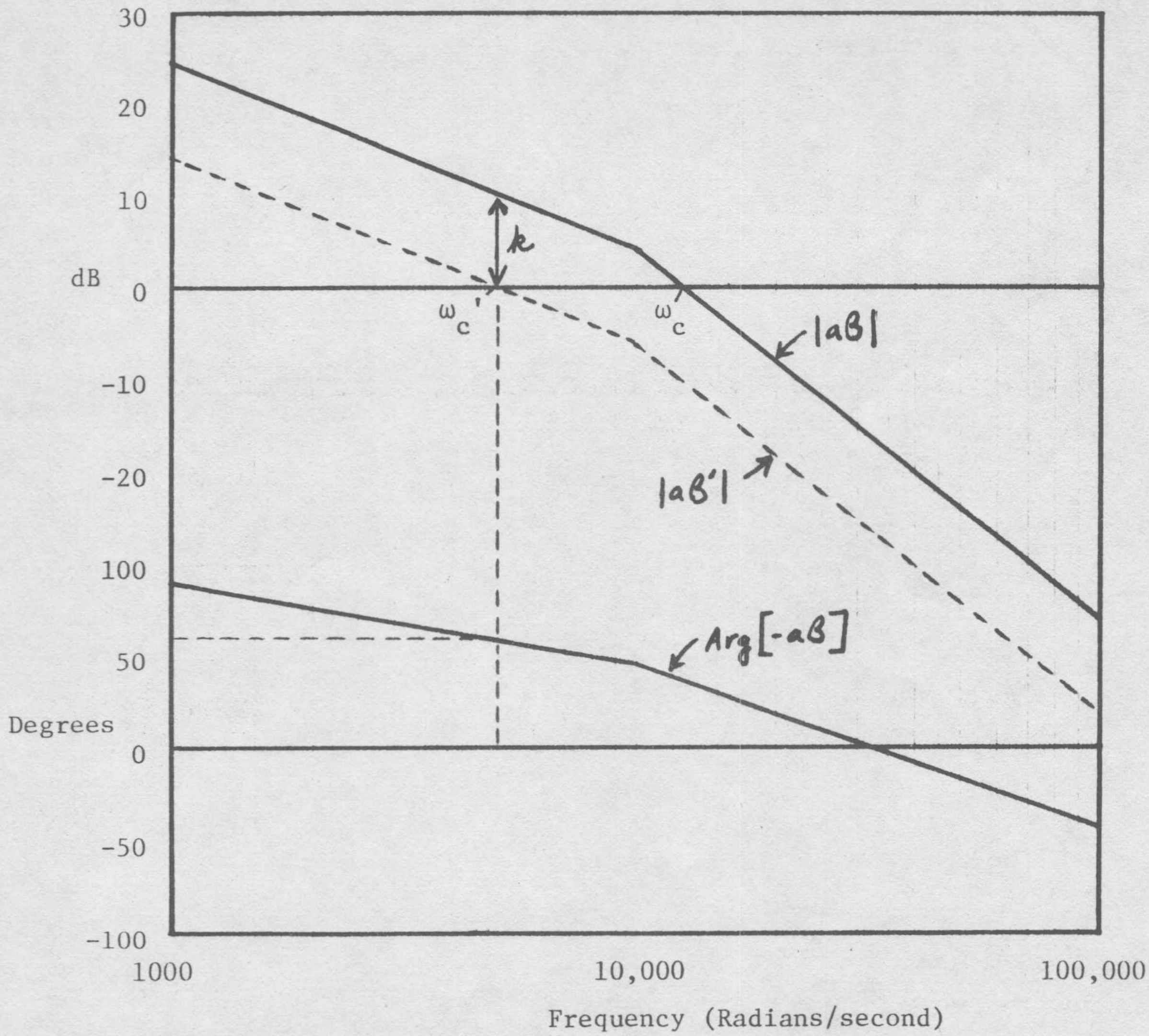


Fig. 5 - Expanded plot of Fig. 4.

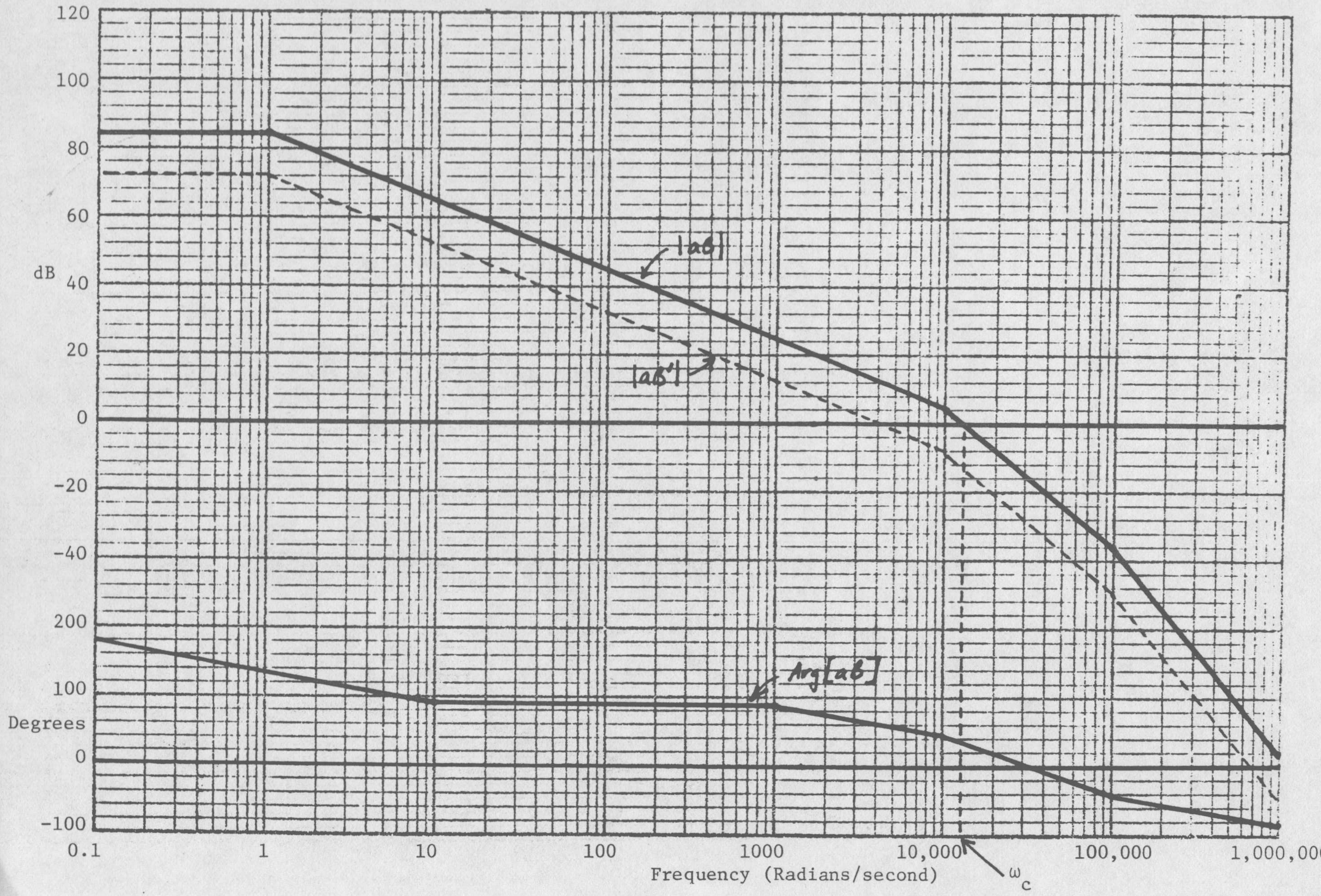


Fig. 4 - Plot of the magnitude and phase of the open loop gain.

The objective of DC loop reduction compensation is to insert a resistor for Z in Fig. 3 and lower the magnitude of the aβ curve until ω_c has been decreased sufficiently so that the desired phase margin is obtained. Consider Fig. 6 which is used to implement this form of compensation.

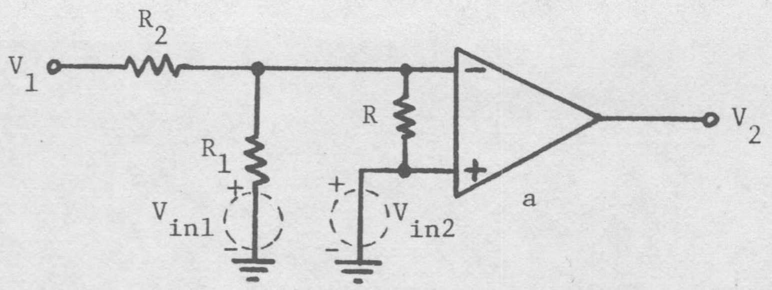


Fig. 6 - DC loop reduction compensation configuration.

One can see from this diagram how the noninverting and inverting configurations become identical under open loop conditions. For the inverting configuration, V_{in1} would be the input voltage source whereas for the noninverting configuration V_{in2} would be the input voltage source. The loop gain of this circuit is found as

$$\frac{V_2}{V_1} = -\ a\beta\left(\frac{1}{1+\beta(R_2/R)}\right) \tag{1}$$

where

$$\beta = \frac{R_1}{R_1+R_2} \tag{2}$$

We see that the β network has been modified by the presence of the resistor R resulting in a new β of

$$\beta' = \beta\left(\frac{1}{1+\beta R_2/R}\right) = K\beta \tag{3}$$

Therefore, if one knows K, then R can be found as

$$R = \frac{\beta R_2}{1/K - 1} \tag{4}$$

Thus, we may outline an approach to DC loop compensation as follows:

1. It is assumed that a and β are known and that ϕ_m is the desired phase margin.

2. Find the frequency, ω_c', where the difference between the phase shift of $-a\beta$ and 0 degrees is equal to ϕ_m. Therefore,

$$\phi_m = \tan^{-1}[a(\omega_c')\beta(\omega_c')] \tag{5}$$

3. Find the difference between $|a\beta|$ at $\omega = \omega_c'$ and zero dB. This is the amount $|a\beta|$ is to be reduced. Let this value be k expressed in dB.

4. Solve for K by

$$K = 10^{-(k/20)} \tag{6}$$

(The minus sign is used because k is a decrease in gain.)

5. Use (4) to solve for the value of R.

<u>Example 1</u>

Use the DC loop reduction method of compensation to achieve a phase margin of 60 degrees for the circuit of Fig. 7 when the amplifier's gain is given as,

$$a(s) = \frac{10^5}{(s+1)(10^{-4}s+1)(10^{-5}s+1)} \tag{7}$$

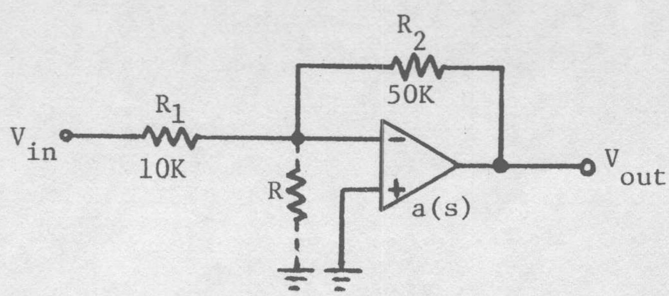


Fig. 7 - Inverting configuration with a closed loop gain of -5.

The value of β=1/6. Therefore $a\beta$ is equal to

$$-a\beta = \frac{(10/6)10^4}{(s+1)(10^{-4}s+1)(10^{-5}s+1)} \tag{8}$$

and is plotted on Figs. 4 and 5. To achieve ϕ_m=60 degrees, ω_c'=5000 rps. At this frequency, k=10 dB. Thus, from (6) K=0.3162. Finally, from (4) we get R = 3.854 KΩ. The new $|a\beta'|$ is plotted as a dotted line on Figs. 4 and 5.

PHASE LAG COMPENSATION

Phase lag compensation is very similar to the DC loop reduction method. However, Z consists of a capacitor placed in series with the resistor R. The phase lag configuration is shown in Fig. 8 where R has been relabelled R_3 and the capacitor is labelled C_3.

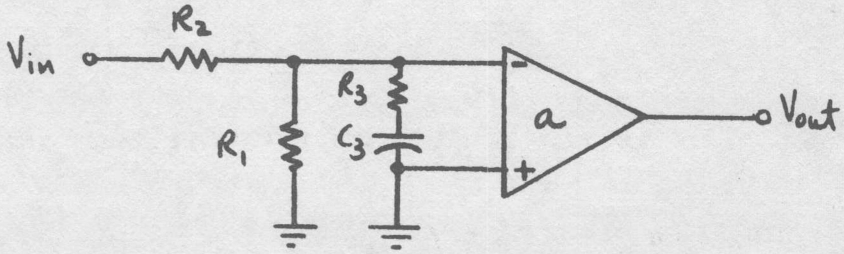


Fig. 8 - Phase lag compensation configuration.

For frequencies above $1/R_3C_3$, Z becomes simply R_3 and we have the same circuit as Fig. 6. However, at frequencies less than $1/R_3C_3$, the reactance of the capacitor dominates R_3 and Z approaches infinity. The advantage of this method is that at low frequencies one has the original aβ curve and therefore better low frequency performance because the factor 1+aβ is larger (i.e. better desensitization) than that for the DC loop reduction scheme.

The transfer function of Fig. 8 can be written as

$$\frac{V_2}{V_1} = a\left(\frac{R_1}{R_1+R_2}\right)\left(\frac{R_1+R_2}{R_1+R_2+\frac{R_1R_2}{R_3}}\right)\left(\frac{s+1/(R_3C_3)}{s+\frac{1}{R_3C_3}\left(\frac{R_1+R_2}{R_1+R_2+(R_1R_2/R_3)}\right)}\right) \tag{9}$$

However, we may express the modified β as,

$$\beta' = \beta\left[1/m\left(\frac{s+\omega_3}{s+(\omega_3/m)}\right)\right] = \beta G_3(s) \tag{10}$$

where

$$\beta = R_1/(R_1+R_2) \tag{11}$$

$$m = \frac{R_1+R_2+(R_1R_2/R_3)}{R_1+R_2} = 1+\beta(R_2/R_3) \tag{12}$$

and

$$\omega_3 = 1/(R_3C_3) \tag{13}$$

The pole-zero plot of (10) is illustrated in Fig. 9. Since m>1, the pole

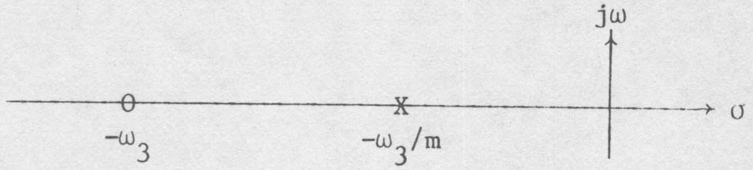
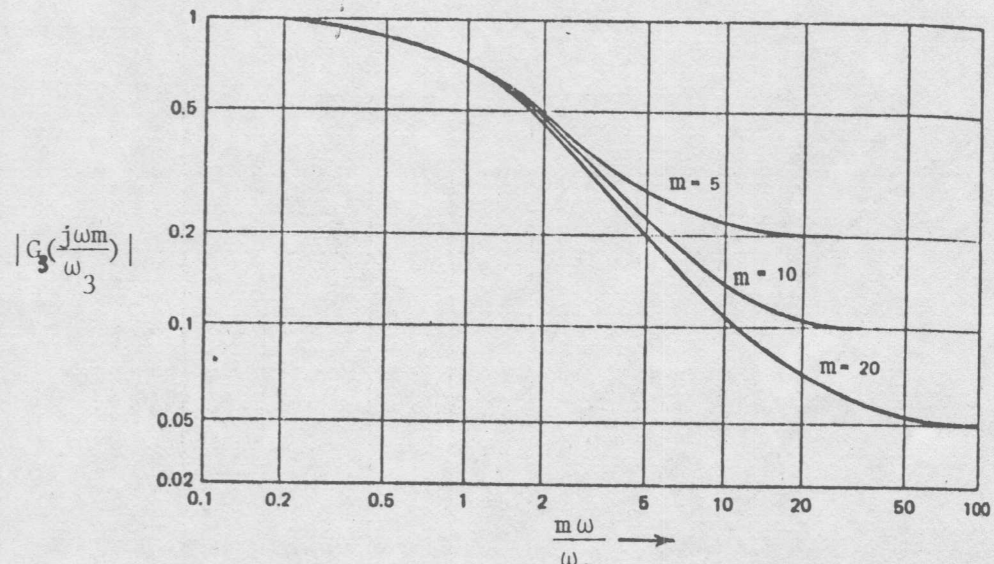
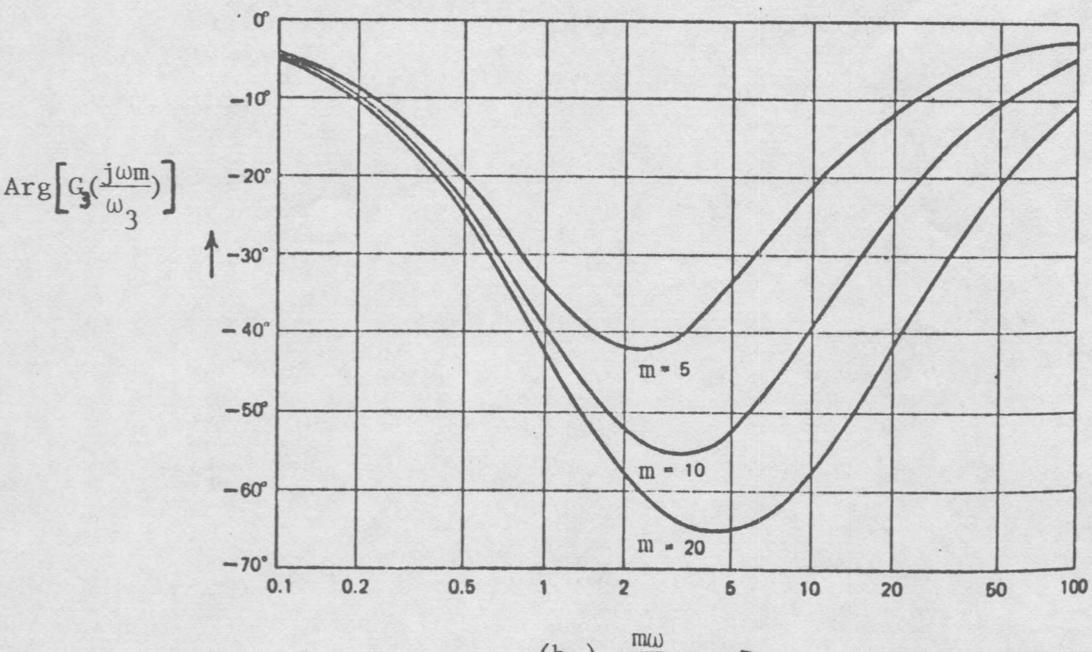


Fig. 9 - Pole-zero plot of phase lag compensation, $G_3(s)$.

is closer to the origin than the zero. A plot of the magnitude and phase of $G_3(s)$ is shown in Fig. 10. We see that the pole break-frequency is located at $\omega/\omega_3=1$ while the break-frequency of the zero is m times larger. We notice



(a.)



(b.) $\frac{m\omega}{\omega_3}$

Fig. 10 - Magnitude and phase response of $G_3(j\omega m/\omega_3)$.

with some concern that the phase shift of $G_3(j\omega m/\omega_3)$ has considerable phase lag at frequencies between the pole and zero breakpoints. This could easily end in disaster if the pole or zero were located close to ω_c. Consequently, in our use of the lag network, we will try to keep the pole and zero sufficiently less than ω_c so that the phase lag contribution of $G_3(j\omega m/\omega_3)$ has almost returned to zero. This consideration leads to one of the rules developed for designing the phase lag compensation network. This rule is to place the zero of the phase lag network at 1/10 of the desired crossover frequency. However, even when the zero is 1/10 of the desired crossover frequency there is still some phase shift of $G_3(j\omega)$. The phase shift of $G_3(j\omega)$ can be expressed in general as,

$$\text{Arg}\left[G_3(j\omega)\right] = \tan^{-1}(\omega/\omega_3) - \tan^{-1}(\omega m/\omega_3) \tag{14}$$

The zero frequency is ω_3 so when the frequency is $10\omega_3$ the phase shift of $G_3(j\omega)$ becomes,

$$\text{Arg}\left[G_3(j\omega)\right] = \tan^{-1}(10) - \tan^{-1}(10m) \tag{15}$$

which can be approximated as

$$\text{Arg}\left[G_3(j\omega)\right] \simeq 84.3^\circ - (90^\circ - 5.73^\circ/m) = -5.71^\circ + 5.73^\circ/m \tag{16}$$

providing that $10m>1$. For values of m in the vicinity of 10, the phase shift due to $G_3(j\omega)$ at $\omega=10\omega_3$ will be approximately -5°. Thus we shall incorporate this additional phase into the following design procedure.

1. It is assumed that a and β are known and that ϕ_m is the desired phase margin.
2. Find the frequency, ω_c', where the difference between the phase of $-a\beta$ and zero degrees is equal to $\phi_m + 5^\circ$.
3. Find the difference between $|a\beta|$ at $\omega = \omega_c'$ and zero dB. This is the amount $|a\beta|$ is to be reduced. Let this value be k expressed in dB.
4. Solve for m by

$$1/m = 10^{-(k/20)} \quad \text{or} \quad m = 10^{(k/20)} \tag{17}$$

5. Using (12) we get

$$R_3 = \beta R_2/(m-1) \tag{18}$$

which is used to solve for R_3.

6. Setting $\omega_3 = \omega_c'/10$ gives

$$C_3 = 10/(\omega_c' R_3) \tag{19}$$

This design procedure will result in the zero of the lag network being placed at a frequency of $\omega_c'/10$ and the pole of the lag network being placed at a frequency of $\omega_c'/10m$.

Example 2

Use the phase lag compensation technique to achieve a phase margin of 60 degrees for the circuit of Fig. 7 when the amplifier's gain is given by (7).

Adding 5 degrees to the desired phase margin gives 65 degrees. The crossover frequency which corresponds to this phase margin is found Fig. 11 as 3500 rps. Thus k = 13 dB. This gives m = 4.4668. Therefore, equations (18) and (19) result in R_3=2.404 KΩ and C_3=1.189μF. The break-points of the phase lag network are 350 rps for the zero and 78.36 rps for the pole. Fig. 11 shows the resulting magnitude and phase plots of the compensated loop gain. It is seen that the phase margin is satisfied. The actual value of the phase margin found by evaluating the arctangents is 64.29 degrees. While this is close enough we see that the reason it is not 65 degrees is that the actual aβ phase at 3500 is 68.72 degrees rather than 65 degrees and that m is small so that the correction factor should be -4.43 degrees rather than -5 degrees. The sum of these two errors equals 4.29 degrees which is exactly the amount of phase shift in excess of ϕ_m. This brings up an important point. If closer accuracy is needed then the asymptotic curves of Fig. 11 should only be used as a guideline and the actual values should be calculated.

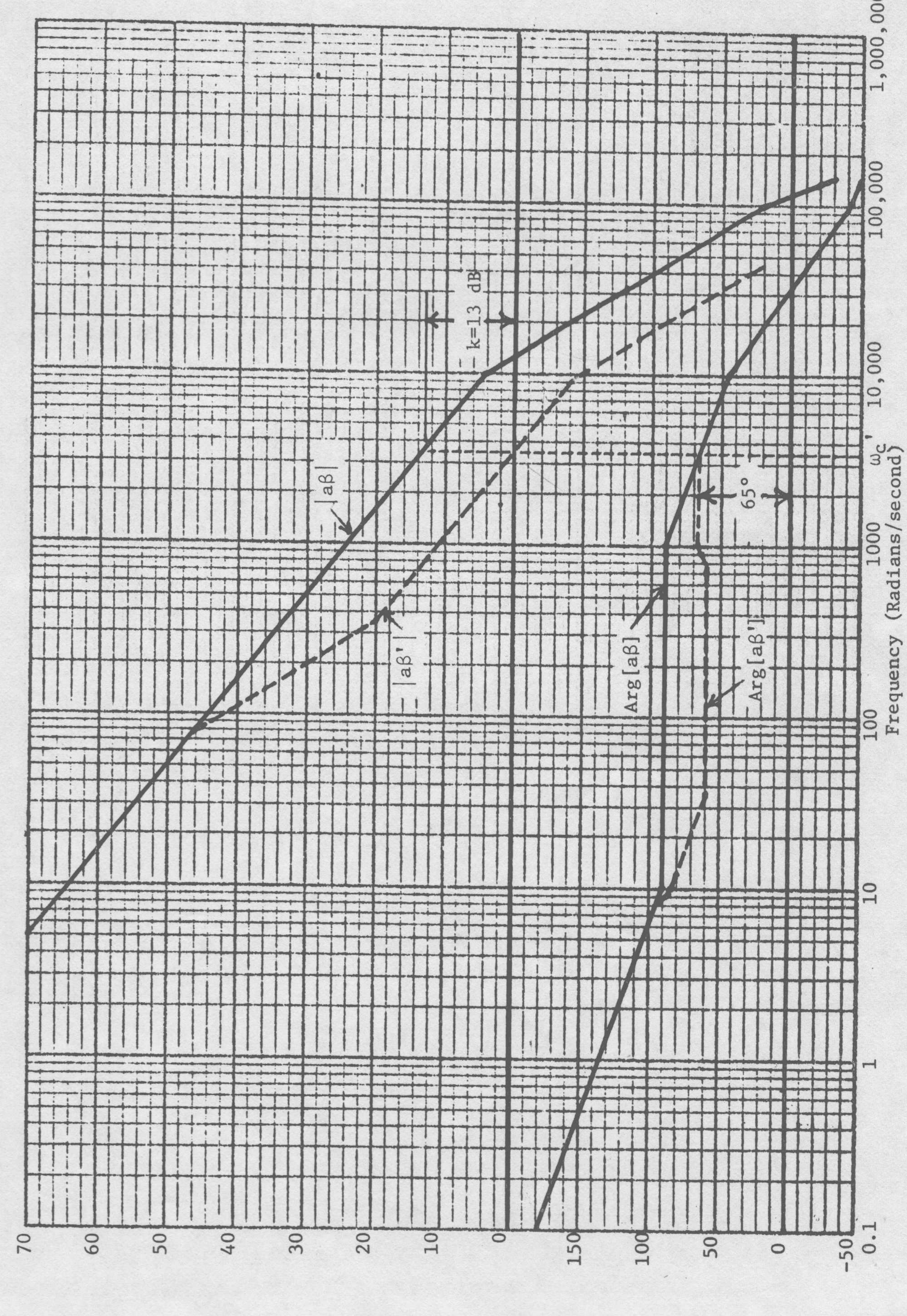


Fig. 11 - Phase lag compensation of Fig. 7.

PHASE LEAD COMPENSATION

Phase lead compensation is similar to the phase lag in that a pole and zero are introduced into the open loop gain. However, in phase lead, the zero is closer to the origin of the complex frequency plane than the pole. Fig. 12 shows a phase lead network. Unfortunately, it is difficult to fit this

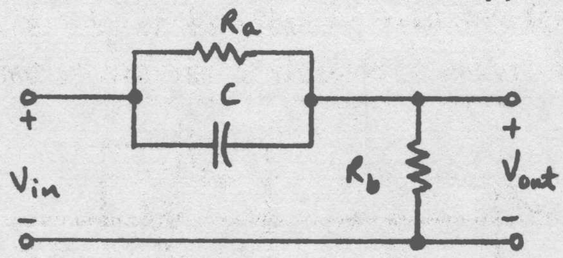


Fig. 12 - A phase lead network.

network into Z of Fig. 3. While there are forms which realize Fig. 12 and these forms only effect the open loop transfer function (See Chapt. 13 of Roberge, Operational Amplifiers, Wiley, 1975) we shall consider a simpler approach. Fig. 13 shows an implementation of phase lead in the open loop circuit of Fig. 3. C_2 is in parallel with the feedback resistor, R_2. R_3 has been included to provide additional flexibility since the values of R_1

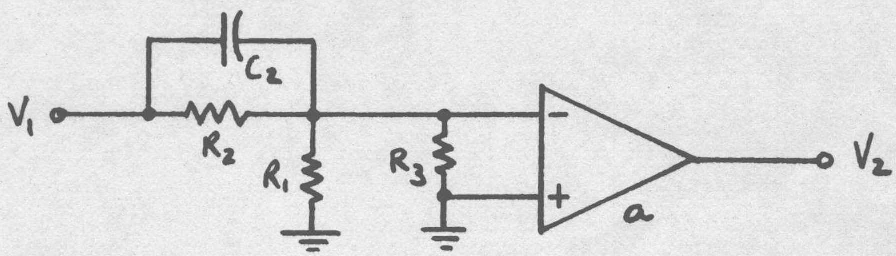


Fig. 13 - Phase lead compensation of Fig. 3.

and R_2 will be determined by the closed loop gain requirements. The value of C_2 will be chosen so that its influence upon the closed loop gain will be minimal.

The open loop gain, V_2/V_1, of Fig. 13 is found as

$$\frac{V_2}{V_1} = a \frac{s + \frac{1}{R_2C_2}}{s + \frac{1}{R_2C_2}\left(1 + \frac{R_2}{R_1} + \frac{R_2}{R_3}\right)} \qquad (20)$$

Therefore, the modified β can be written as

$$\beta' = \frac{s + \frac{1}{R_2C_2}}{s + \frac{1}{R_2C_2}\left(1 + \frac{R_2}{R_3} + \frac{R_2}{R_1}\right)} = \frac{s + \omega_2/m}{s + \omega_2} \tag{21}$$

where

$$\omega_2 = \frac{1}{R_2C_2}\left(1 + \frac{R_2}{R_1} + \frac{R_2}{R_3}\right) \tag{22}$$

and

$$m = 1 + \frac{R_2}{R_1} + \frac{R_2}{R_3} = \frac{1}{\beta} + \frac{R_2}{R_3} \tag{23}$$

Note that if R_3 is infinite, that

$$\omega_2 = 1/(\beta R_2C_2) \tag{24}$$

and

$$m = 1/\beta \tag{25}$$

Equation (21) is the general form of a phase lead network transfer function. Let us designate this transfer function in (21) as $G_2(s)$. The pole-zero plot of $G_2(s)$ is shown in Fig. 14.

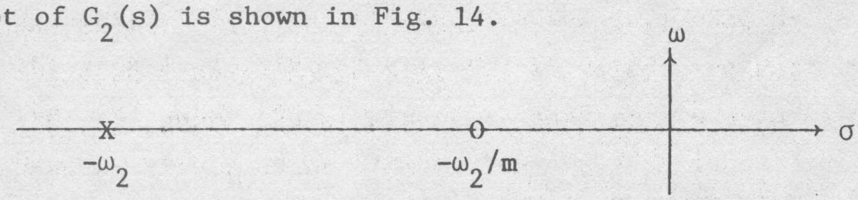


Fig. 14 - Pole-zero plot of a phase lead network.

The magnitude and phase response for various values of m is shown in Fig. 15. It is seen that the pole is located at $\omega/\omega_2=1$ while the zero is at $\omega/\omega_2=1/m$. We see that the phase lead network is useful for counteracting the phase lag of the amplifier. However, we must be careful that the phase lead network does not "boost" the open loop gain and remove the stabilizing influence of the leading phase shift. In order to minimize the influence of the magnitude of $G_2(s)$ upon the open loop gain, we will try to select the zero and pole of $G_2(s)$ so that they are above the crossover frequency. Therefore, any magnitude boosting will have no effect on stability when the open loop gain is less than unity. Unfortunately, we can only partially achieve these results and the phase lead design method will by nature have to be iterative in order to correct for the effects of the open loop magnitude boosting.

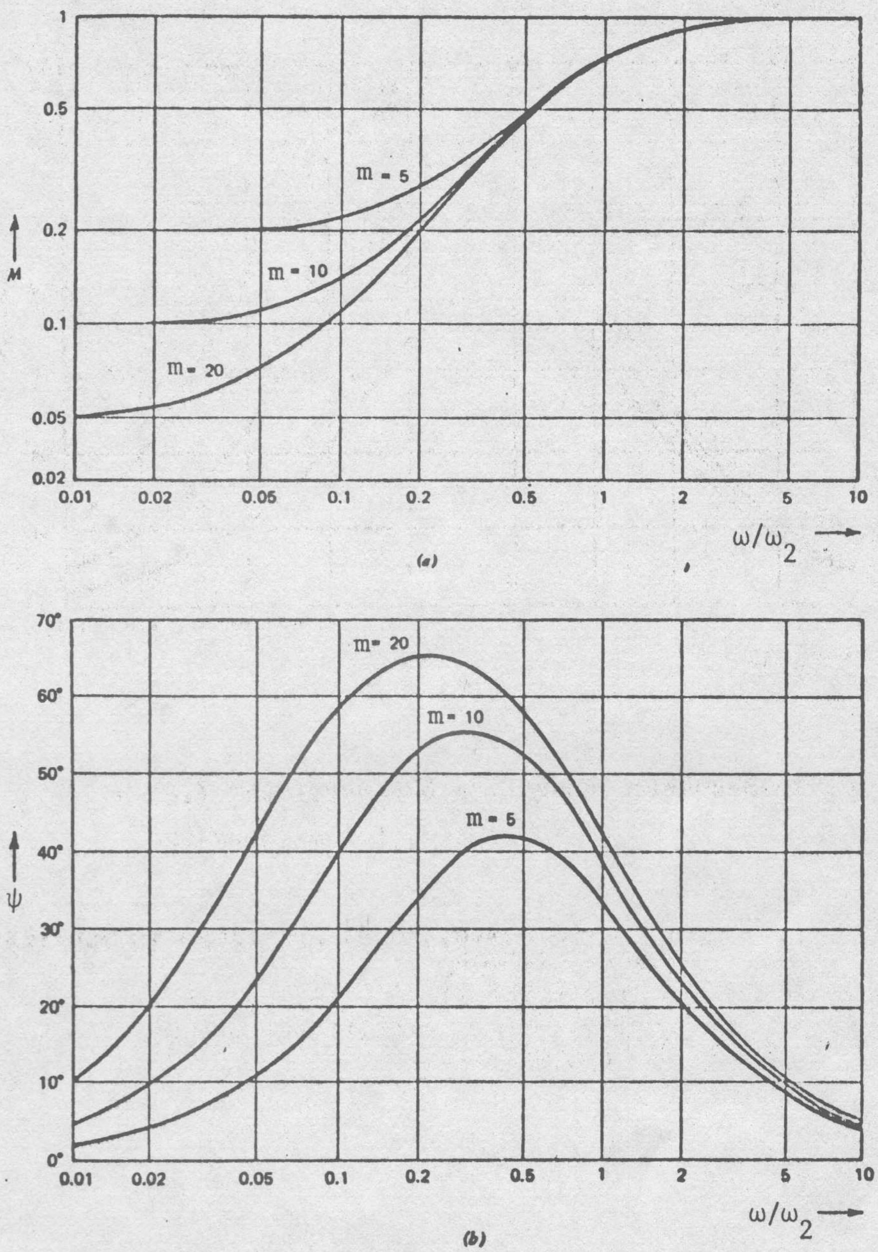


Fig. 15 - (a.) Magnitude and (b.) phase response of the phase lead network, $G_2(s)$.

The magnitude of $G_2(s)$ can be expressed as

$$|G_2(j\omega)| = (1/m)\sqrt{\frac{1 + (\omega m/\omega_2)^2}{1 + (\omega/\omega_2)^2}} \qquad (26)$$

Similarily, the phase of $G_2(s)$ is given as

$$\psi(j\omega) = \mathrm{Arg}[G_2(j\omega)] = \tan^{-1}\left(\frac{\omega m}{\omega_2}\right) - \tan^{-1}\left(\frac{\omega}{\omega_2}\right) = \tan^{-1}\left[\frac{\frac{\omega m}{\omega_2} - \frac{\omega}{\omega_2}}{1 + m(\omega/\omega_2)^2}\right] \qquad (27)$$

It is of interest to note that the maximum phase of $\psi(j\omega)$ is equal to

$$\psi_m(j\omega_m) = \tan^{-1}\left(\frac{m-1}{2\sqrt{m}}\right) = \sin^{-1}\left(\frac{m+1}{m-1}\right) \qquad (28)$$

which occurs at a frequency of

$$\omega_m = \omega_2/\sqrt{m} \qquad (29)$$

The magnitude of $G_2(j\omega)$ at ω_m is given as

$$|G_2(j\omega_m)| = 1/\sqrt{m} \qquad (30)$$

The design approach for the phase lead case is more complicated than the previous two methods. The reason is that m is not a variable and is generally given by (25), when R_3 is equal to infinity, and that the amount of phase lead available from the phase lead network is limited and may not be sufficient to meet the phase margin specifications. Therefore, the first step in our design procedure will be to find where we are in relation to these two problems.

Let us first assume that m is fixed and find out whether the phase lead compensation scheme can meet the specifications. Let the phase margin of the uncompensated open loop gain be PM. If the desired phase margin is ϕ_m, then the amount of phase lead required for compensation is

$$\psi = \phi_m - PM + \theta \qquad (31)$$

where θ is a phase correction factor which accounts for magnitude boosting of the open loop gain and a subsequent shift of ω_c to the right to a new value ω_c'. Unfortunately, θ is not known prior to the problem but will generally fall in the range of 3 to 10 degrees for most problems. Next, we can find the location of the pole, ω_2, and the zero, ω_2/m, breakpoints of the phase lead network by equating $\psi(j\omega_i)$ given in (27) to (31). Solving for ω_2/ω_i yields,

$$\frac{\omega_2}{\omega_i} = \frac{m-1}{2\tan\psi}\left[1 + \sqrt{1 - \frac{4m}{(m-1)^2}\tan^2\psi}\right] \qquad (32)$$

This expression shows where to locate the pole of the phase lead network with respect to a frequency ω_i. When ω_2/ω_i becomes equal to $\sqrt{m}$, then we see from (29) that the maximum possible phase lead is obtained from the phase lead network. For smaller values of ω_2/ω_i than $\sqrt{m}$, less phase lead will be obtained. Therefore, we may write that

$$\omega_2/\omega_i > \sqrt{m} \tag{33}$$

in order to satisfy the phase margin specifications. One can use this constraint to solve for the maximum value of ψ. However, a more stringent constraint is obtained from the requirement that ω_2/ω_i must be a real number. We find that for the argument of the radical of (32) to be positive that

$$\psi < \tan^{-1}\left[\frac{m-1}{2\sqrt{m}}\right] \tag{34}$$

If (34) cannot be satisfied, then one will have to be able to vary m or be able to use a different form of compensation as well as phase lead compensation to meet the phase margin requirements. A design procedure based on the above ideas is outlined below:

$\psi < \tan^{-1}[(m-1)/\sqrt{m}\,]$

1. It assumed that a and β are known, ϕ_m is specified, R_3 is infinite, and that $m = 1/\beta$.
2. Choose a value for θ. (θ = 5 degrees is a good first guess.)
3. Find ψ from (31).
4. Solve for ω_2 and ω_2/m in terms of ω_i.
5. Evaluate the parameter K where K is defined as

$$K = \frac{|\beta'|}{|\beta|} = \sqrt{\frac{1 + (\omega_i\omega_m/\omega_2)^2}{1 + (\omega_i/\omega_2)^2}} \tag{35}$$

6. Find the frequency, ω_c', where the uncompensated open loop gain is equal to $-20 \log_{10}K$. (We will set ω_i equal to this frequency.)
7. Find the phase margin, PM', at ω_c'.
8. Form $\theta' = PM - PM'$.
9. a.) If $\theta' > \theta$, increase the value of θ and go back to step 3.
   b.) If $\theta' < \theta$, decrease the value of θ and go back to step 3.
   c. If $\theta' \cong \theta$, then the value of C_2 is given by (24) and (25) where ω_2 is obtained from step 4 with $\omega_i = \omega_c'$. The new crossover frequency is ω_c'.

The above design procedure is illustrated by the following example.

Example 3

Use the phase lead compensation technique to achieve a phase margin of 60 degrees for the circuit of Fig. 7 when the amplifier's gain is given by (7).

Evaluating (31) with $\theta=0°$ gives $\psi=23°$. With $m = 1/\beta = 6$, (34) gives the constraint that $\psi<45.6°$. Therefore, this problem should be easily solved by the previous design procedure providing θ does not become larger than about 22°. The iterative procedure is started with $\theta=0°$. $\omega_2/\omega_i=11.2457$ and $\omega_2/\omega_i m$ is 1.8743. This gives K=1.1290 which corresponds to a -1.0537 dB boost. Thus $\omega_c'=1.32\times10^4$ rps which gives PM'= 35°. Since PM=37°, then θ' is about 2°.

The second time through the procedure, $\theta=3°$. Thus $\omega_2/\omega_i=9.6284$ and $\omega_2/\omega_i m=1.6047$ which gives a K=1.1720. This value of K corresponds to a 1.3783 dB boost which leads to $\omega_c'=1.36\times10^4$ rps. The phase margin at this frequency is PM'= 34°. Thus $\theta'=3$. Now we set $\omega_i=\omega_c'=1.36\times10^4$ rps. This gives the zero breakpoint at 2.182×10^4 rps and the pole breakpoint at 13.0×10^4 rps. The new crossover frequency is 1.36×10^4 rps which has shifted to the right from the original crossover frequency of 1.25×10^4 rps. The resulting phase margin is indicated upon Fig. 16 as ϕ_m. We note that the asymptotic value exceeds 60 degrees. However, the actual value as calculated from the sum of the arctangents is 54.54°. One of the problems in our technique is that we used the phase margins (PM and PM') from the asymptotic curves rather than the exact values. If more accuracy is desired then one should use the actual values of phase margin. This was done and it was found that the value of $\theta=6°$, resulted in a new crossover frequency of 1.48×10^4 rps and an actual phase margin of 60.1024°. For the more accurate design, the last iteration gave $\omega_2=6.3363\omega_i=6.3363\omega_c'=9.3777\times10^4$ rps. Thus from (24) we find that $C_2=1.28nF$. Happily, we note that the influence of C_2 upon the closed loop transfer function will occur beyond the new crossover frequency.

The design procedure listed above may be unsuccessful if the value of ψ becomes too great. Therefore, we need to make both m and ω_2 variables that can be used for design purpose. Even then, there will be a limit to what the phase lead compensation method can accomplish. At this point it will be necessary to augment the phase lead technique by one of the previous two techniques.

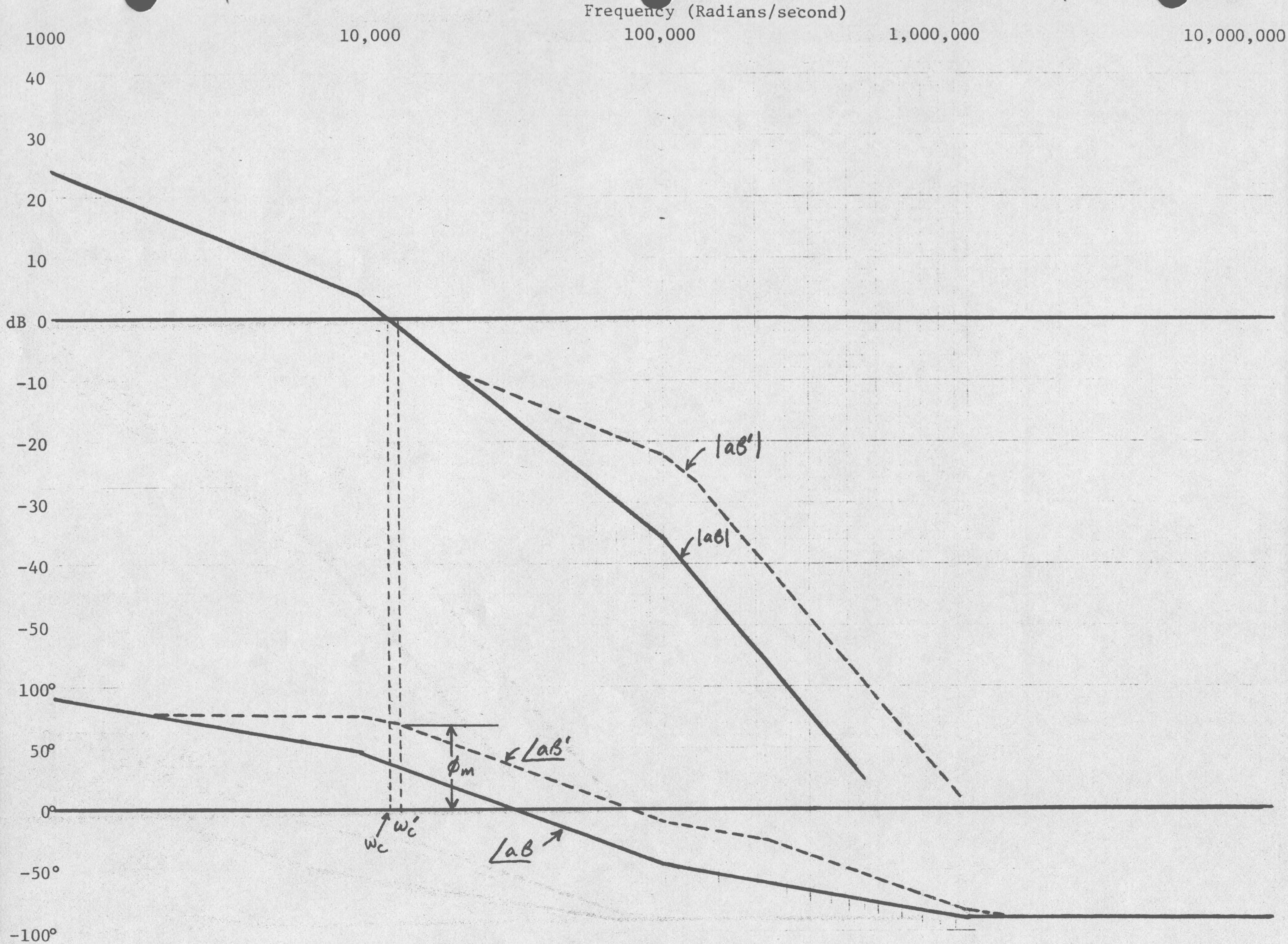


Fig. 16 - Lead compensation example.

$R_3 \neq$ infinity

1. Reduce the desired phase margin ϕ_m to a value ϕ_m' which will satisfy (34). Use the previous design procedure to find ω_2.
2. Find the magnitude of the phase lead compensated open loop gain in step 1 where the additional phase margin $\phi_m - \phi_m'$ is obtained. Designate this value as k in dB. Use the DC loop reduction technique to achieve the additional phase margin. R_3 can be found from

$$R_3 = \frac{\beta R_2}{(1/K)-1} \tag{36}$$

where K is given by (6).
3. Finally, C_2 is designed from (22) as

$$C_2 = \frac{1}{R_2\omega_2}\left(\beta + \frac{R_2}{R_3}\right) \tag{37}$$

An example is given to illustrate this method.

Example 4

Let us assume that the amplifier of Example 3 has 10 dB additional gain and let us attempt to compensate the circuit in Fig. 7 using this amplifier to have a phase margin of 60°. The poles of the amplifier are the same as given in (7).

The open loop magnitude and phase of the circuit is shown in Fig. 17. We note that the phase margin, PM, is about 12° at a crossover frequency of about $2.2\text{x}10^4$ rps. We see that ψ is equal to 48° assuming θ equal 0°. Eq. (34) shows that the maximum value of ψ is about 45.5°. Therefore we cannot expect to obtain the desired phase margin using the previous design. Thus let us select a phase margin, ϕ_m', of 45°. Using the iterative design on page 16 and using the exact phase shifts rather than the asymptotic values, we find that for θ=9°that ω_2=$11.843\text{x}10^4$ rps, ω_2/m=$1.9738\text{x}10^4$ rps, ω_c'=$2.9\text{x}10^4$ rps, and ϕ_m'=44.86°.

The resulting partial compensation by the phase lead technique is shown on Fig. 17. Now we need to find the frequency where we obtain the additional 15° of phase margin. Unfortunately, the phase asymptotes are treacherous in the vicinity of a breakpoint so that we again use the exact

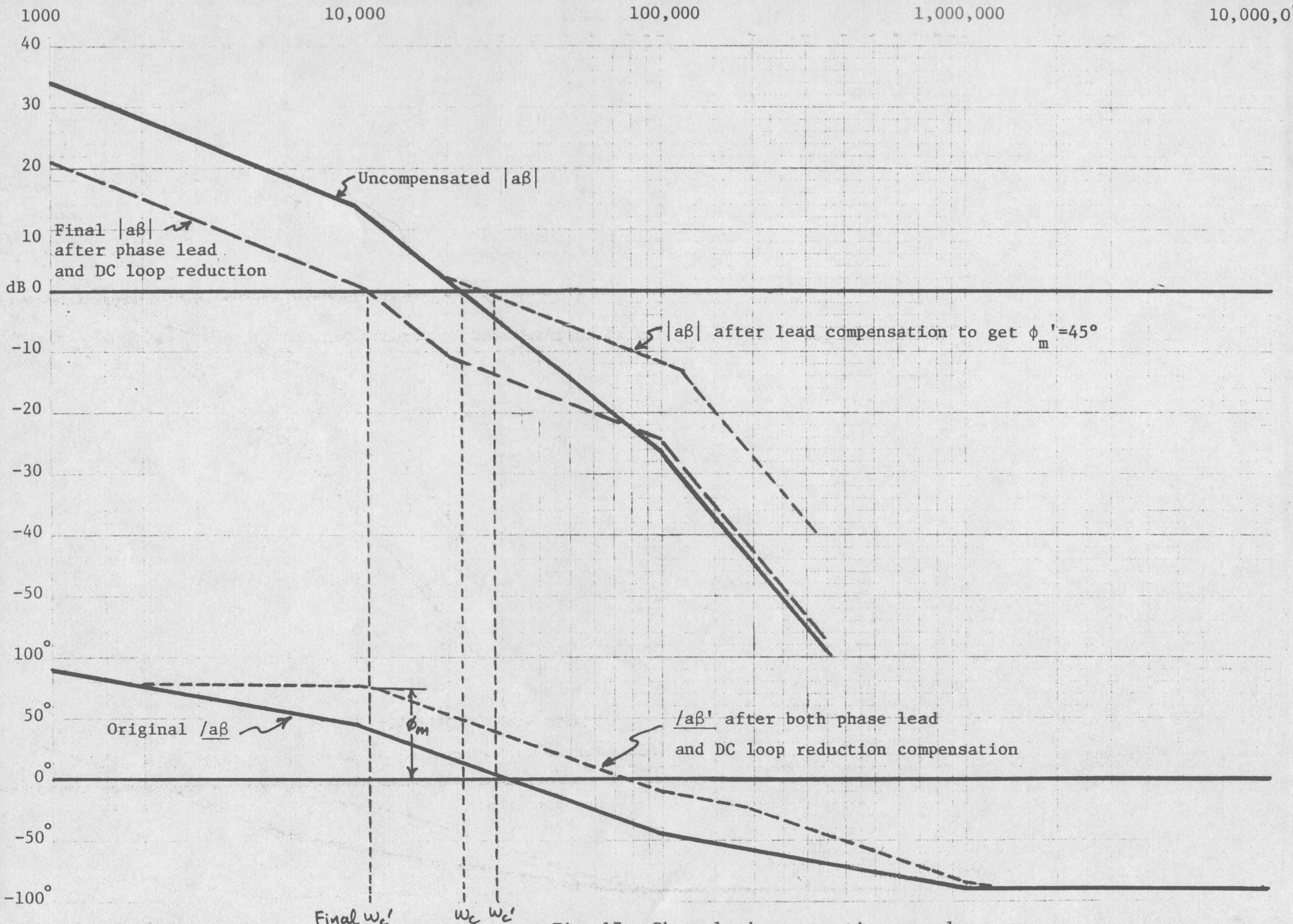


Fig. 17 - Phase lead compensation example.

phase shifts determined by the sum of the arctangents of the breakpoints. Therefore, we find that at $\omega_c'=1.1x10^4$ rps that the actual phase shift is 59.8°. At this frequency k is equal to about 12 dB. Hence, K=0.2512. Eq. (36) gives $R_3=2.795K\Omega$ and (37) gives $C_2=4.033nF$. The resulting compensated open loop magnitude and phase response is shown on Fig. 17.